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CHEBYSHEV'S BIAS AND GENERALIZED RIEMANN HYPOTHESIS

ADEL ALAHMADI[1], **MICHEL PLANAT**[2] **and PATRICK SOLÉ**[3]

[1]MECAA
King Abdulaziz University
Jeddah
Saudi Arabia

[2]Institut FEMTO-ST
CNRS, 32 Avenue de l'Observatoire
F-25044 Besançon
France
e-mail: michel.planat@femto-st.fr

[3]Telecom Paristech
46 rue Barrault
75634 Paris Cedex 13
France

Abstract

The oscillations of the prime counting function $\pi(x)$ around the logarithmic integral $\mathrm{li}(x)$ are known to be controlled by the zeros of Riemann's zeta function $\zeta(s)$. Similarly, the discrepancy $\pi(x; q, R) - \pi(x; q, N)$ between the number of primes modulo q in a quadratic residue class R and in a quadratic nonresidue class N-the so-called Chebyshev's bias- is controlled by the zeros of a Dirichlet

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L-function $L(s, q)$ with the modulus q. In this work, we introduce a new bias, called the regularized Chebyshev's bias, whose non-negativity is expected to be equivalent to a Riemann hypothesis for $L(s, q)$. In particular, under the generalized Riemann hypothesis, this new bias should be positive for all integers q. The results are motivated and illustrated by extensive numerical calculations.

1. Introduction

In the following, we denote by $\pi(x)$ the prime counting function and by $\pi(x; q, a)$ the number of primes not exceeding x and congruent to a mod q. The asymptotic law for the distribution of primes is the prime number theorem $\pi(x) \sim \frac{x}{\log x}$. Correspondingly, one gets [5, Equation (14), p. 125]

$$\pi(x; q, a) \sim \frac{\pi(x)}{\phi(q)}, \tag{1.1}$$

that is, one expects the same number of primes in each residue class a mod q, if $(a, q) = 1$. Chebyshev's bias is the observation that, contrarily to expectations, $\pi(x; q, N) > \pi(x; q, R)$ most of the times, when N is not a square modulo q, but R is.

Let us start with the bias

$$\delta(x, 4) := \pi(x; 4, 3) - \pi(x; 4, 1), \tag{1.2}$$

found between the number of primes in the quadratic nonresidue class $N = 3 \mod 4$ and the number of primes in the quadratic residue class $R = 1 \mod 4$. The values $\delta(10^n, 4)$, $n \geq 1$, form the increasing sequence

$$A091295 = \{1, 2, 7, 10, 25, 147, 218, 446, 551, 5960, \ldots\}.$$

The bias is found to be negative in thin zones of size

$$\{2, 410, 15\ 358, 41346, 42\ 233\ 786, 416\ 889\ 978, \ldots\},$$

spread over the location of primes of maximum negative bias [1]

$$\{26861, 623\,681, 12\,366\,589, 951\,867\,937, 6\,345\,026\,833, 18\,699\,356\,321, \ldots\}.$$

It has been proved that there are infinitely many sign changes in the Chebyshev's bias (1.2). This follows from the Littlewood's oscillation theorem [6, 8]

$$\delta(x,\ 4) := \Omega_{\pm}\left(\frac{x^{1/2}}{\log x}\log_3 x\right). \tag{1.3}$$

A useful measure of the Chebyshev's bias is the logarithmic density [6, 7, 13]

$$d(A) = \lim_{x\to\infty}\frac{1}{\log x}\sum_{a\in A,\, a\leq x}\frac{1}{a}, \tag{1.4}$$

for the positive Δ^{+} and negative Δ^{-} regions calculated as $d(\Delta^{+}) = 0.9959$ and $d(\Delta^{-}) = 0.0041$.

In essence, Chebyshev's bias $\delta(x,\ 4)$ is similar to the bias

$$\delta(x) := \mathrm{Li}(x) - \pi(x). \tag{1.5}$$

It is known that $\delta(x) > 0$ up to the (very large) Skewes' number $x_1 \sim 1.40\times 10^{316}$ but, according to Littlewood's theorem, there are also infinitely many sign changes of $\delta(x)$ [8].

The reason why the asymmetry in (1.5) is so much pronounced is encoded in the following approximation of the bias [3, 13][1]:

$$\delta(x) \sim \frac{\sqrt{x}}{\log x}\left(1 + 2\sum_{\gamma}\frac{\sin(\gamma\log x + \alpha_{\gamma})}{\sqrt{1/4 + \gamma^2}}\right), \tag{1.6}$$

where $\alpha_{\gamma} = \cot^{-1}(2\gamma)$ and γ is the imaginary part of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. The smallest value of γ is quite large, $\gamma_1 \sim 14.134$, and leads to a large asymmetry in (1.5).

[1] The bias may also be approached in a different way by relating it to the second order Landau-Ramanujan constant [10].

Under the assumption that the generalized Riemann hypothesis (GRH) holds that is, if the Dirichlet L-function with non-trivial real character κ_4

$$L(s, \kappa_4) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^s}, \tag{1.7}$$

has all its non-trivial zeros located on the vertical axis $\Re(s) = \frac{1}{2}$, then the formula (1.6) also holds for the Chebyshev's bias $\delta(x, 4)$. The lowest non-trivial zero of $L(s, \kappa_4)$ is at the ordinate $\gamma_1 \sim 6.02$, a much smaller value than the one corresponding to $\zeta(s)$, so that the bias is also much smaller.

A second factor controls the aforementioned asymmetry of a L-function of real non-trivial character κ, it is the *variance* [9]

$$V(\kappa) = \sum_{\gamma > 0} \frac{2}{1/4 + \gamma^2}. \tag{1.8}$$

For the function $\zeta(s)$ and $L(s, \kappa_4)$, one gets $V = 0.045$ and $V = 0.155$, respectively.

Our main goal. In a groundbreaking paper, Robin reformulated the unconditional bias (1.5) as a conditional one involving the second Chebyshev function $\psi(x) = \sum_{p^k \leq x} \log p$.

The equality $\delta'(x) := \mathrm{li}[\psi(x)] - \pi(x) > 0$ is equivalent to RH. (1.9)

This statement is given as Corollary 1.2 in [11] and led the second and third author of the present work to derive a *good prime counting function*

$$\pi(x) = \sum_{n=1}^{3} \mu(n) \mathrm{li}[\psi(x)^{1/n}]. \tag{1.10}$$

Here, we are interested in a similar method to *regularize* the Chebyshev's bias in a conditional way similar to (1.9). In [12], Robin introduced the function

$$B(x;\, q,\, a) = \mathrm{li}[\phi(q)\psi(x;\, q,\, a)] - \phi(q)\pi(x;\, q,\, a), \tag{1.11}$$

that generalizes (1.9) and applies it to the residue class a mod q, with $\psi(x;\, q,\, a)$ the generalized second Chebyshev's function. Under GRH, he proved that [12, Lemma 2, p. 265]

$$B(x;\, q,\, a) = \Omega_{\pm}\left(\frac{\sqrt{x}}{\log^2 x}\right), \quad x \to \infty, \tag{1.12}$$

that is,

$$\text{The inequality } B(x;\, q,\, a) > 0 \text{ is equivalent to GRH.} \tag{1.13}$$

For the Chebyshev's bias, we now need a proposition taking into account two residue classes such that $a = N$ (a quadratic nonresidue) and $a = R$ (a quadratic one).

Proposition 1.1. *Let* $B(x;\, q,\, a)$ *be the Robin B-function defined in* (1.11), *and* R (*resp.*, N) *be a quadratic residue modulo* q (*resp.*, *a quadratic nonresidue*), *then the statement* $\delta'(x,\, q) := B(x;\, q,\, R) - B(x;\, q,\, N) > 0$, $\forall x$ (i), *is equivalent to GRH for the modulus* q.

The present paper deals about the numerical justification of Proposition 1.1 in Section 2 and its tentative proof in Section 3. The calculations are performed with the software Magma [4] available on a 96MB segment of the cluster at the University of Franche-Comté.

2. The Regularized Chebyshev's Bias

All over this section, we are interested in the prime champions of the Chebyshev's bias $\delta(x,\, q)$ (as defined in (1.2) or (2.3), depending on the context). We separate the prime champions leading to a positive/negative bias. Thus, the n-th prime champion satisfies

$$\delta^{(\epsilon)}(x_n,\, q) = \epsilon n, \quad \epsilon = \pm 1. \tag{2.1}$$

We also introduce a new measure of the overall bias $b(q)$, dedicated to our plots, as follows:

$$b(q) = \sum_{n, \epsilon} \frac{\delta^{(\epsilon)}(x_n, q)}{x_n}. \tag{2.2}$$

Indeed, smaller is the bias lower is the value of $b(q)$. Anticipating over the results presented below, Table 1 summarize the calculations.

Table 1. The new bias (2.2) (column 2) and the standard logarithmic density (1.4) (column 3)

Modulus q	Bias $b(q)$	Log density $d(\Delta^+)$	First zero γ_1
4	0.7926	0.9959 [3]	14.134
11	0.1841	0.9167 [3]	0.2029
13	0.2803	0.9443 [3]	3.119
163	0.0809	0.55 [9]	2.477

Chebyshev's bias for the modulus $q = 4$. As explained in the Introduction, our goal in this paper is to reexpress a standard Chebyshev's bias $\delta(x, q)$ into a regularized one $\delta'(x, q)$, that is always positive under the condition that GRH holds. Indeed, we do not discover any numerical violation of GRH and we always obtain a positive $\delta'(x, q)$. The asymmetry of Chebyshev's bias arises in the plot δ vs δ', where the fall of the normalized bias $\frac{\delta}{\sqrt{x}}$ is faster for negative values of δ than for positive ones. Figure 1 clarifies this effect for the historic modulus $q = 4$. We restricted our plot to the champions of the bias δ and separated positive and negative champions.

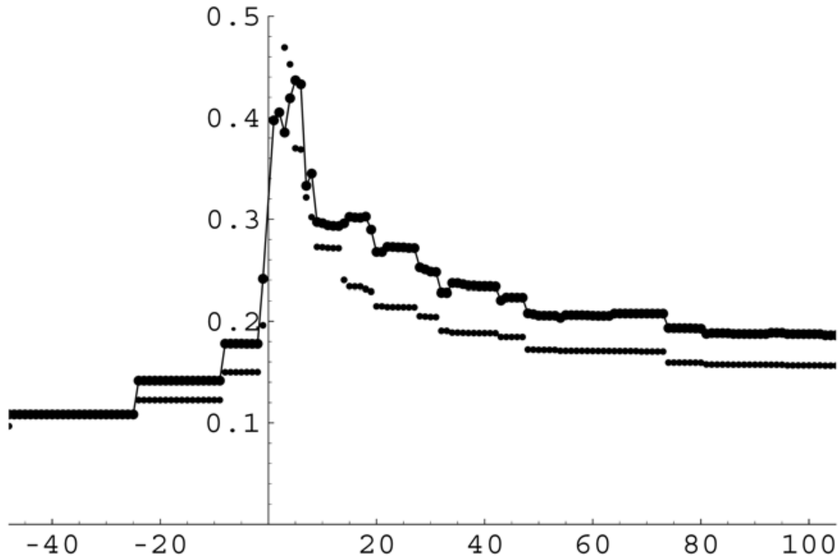

Figure 1. The normalized regularized bias $\delta'(x, 4)/\sqrt{x}$ versus the Chebyshev's bias $\delta(x, 4)$ at the prime champions of $\delta(x, 4)$ (when $\delta(x, 4) > 0$) and at the prime champions of $-\delta(x, 4)$ (when $\delta(x, 4) < 0$). The extremal prime champions in the plot are $x = 359327$ (with $\delta = 105$) and $x = 951867937$ (with $\delta = -48$). The curve is asymmetric around the vertical axis, a fact that reflects the asymmetry of the Chebyshev's bias. As explained in the text, a violation of GRH would imply a negative value of the regularized bias $\delta'(x, 4)$. The small dot curve corresponds to the fit of $\delta'(x, 4)/\sqrt{x}$ by $2/\log x$ calculated in Section 3.

Chebyshev's bias for a prime modulus p**.** For a prime modulus p, we define the bias so as to obtain an averaging over all differences $\pi(x;\ p,\ N) - \pi(x;\ p,\ R)$, whereas above N and R denote a quadratic nonresidue and a quadratic residue, respectively,

$$\delta(x,\ p) = -\sum_{a} \left(\frac{a}{p}\right) \pi(x;\ p,\ a), \tag{2.3}$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol. Correspondingly, we define the regularized bias as

$$\delta'(x,\ p) = \frac{1}{\lfloor p / 2 \rfloor} \sum_{a} \left(\frac{a}{p}\right) B(x;\ p,\ a). \tag{2.4}$$

Proposition 2.1. *Let p be a selected prime modulus and* $\delta'(x,\ p)$ *as in* (2.4), *then the statement* $\delta'(x,\ p) > 0,\ \forall x,$ *is equivalent to GRH for the modulus p.*

As mentioned in the Introduction, the Chebyshev's bias is much influenced by the location of the first non-trivial zero of the function $L(s, \kappa_q)$, κ_q being the real non-principal character modulo q. This is especially true for $L(s, \kappa_{163})$ with its smaller non-trivial zero at $\gamma \sim 0.2029$ [3]. The first negative values occur at $\{15073, 15077, 15083, \ldots\}$.

Figure 2 represents the Chebyshev's bias δ' for the modulus $q = 163$ versus the standard one δ (thick dots). That asymmetry of the Chebyshev's bias is revealed at small values of $|\delta|$, where the fit of the regularized bias by the curve $2 / \log x$ is not good (thin dots).

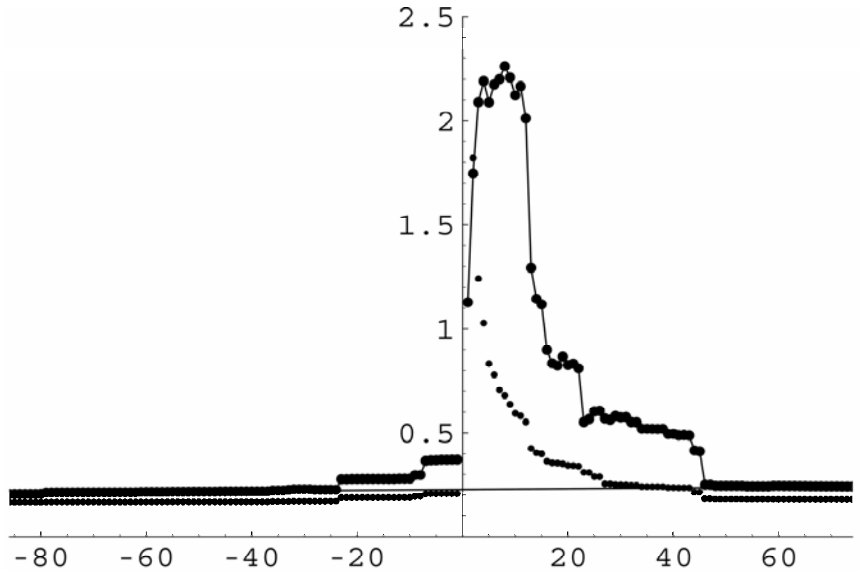

Figure 2. The normalized regularized bias $\delta'(x, 163)/\sqrt{x}$ versus the Chebyshev's bias $\delta(x, 163)$ at all the prime champions of $|\delta(x, 163)|$ [from $|\delta(x, 163)| > 74$, the bias is $\delta(x, 163) < 0$ negative], superimposed to the curve at the prime champions of $-\delta(x, 163)$ (when $\delta(x, 163) < 0$). The extremal prime champions in the plot are $x = 68491$ (with $\delta = 74$) and $x = 174637$ (with $\delta = -86$). The asymmetry is still clearly visible in the range of small values of $|\delta|$, but tends to disappear in the range of high values of $|\delta|$. The small dot curve corresponds to the fit of $\delta'(x, 163)/\sqrt{x}$ by $2/\log x$ calculated in Section 3.

For the modulus $q = 13$, the imaginary part of the first zero is not especially small, $\gamma_1 \sim 3.119$, but the variance (1.8) is quite high, $V(\kappa_{-13}) \sim 0.396$. The first negative values of $\delta(x, 13)$ at primes occur when $\{2083, 2089, 10531, \ldots\}$. Figure 3 represents the Chebyshev's bias δ' for the modulus $q = 13$ versus the standard one δ (thick dots) as compared to the fit by $2/\log x$ (thin dots).

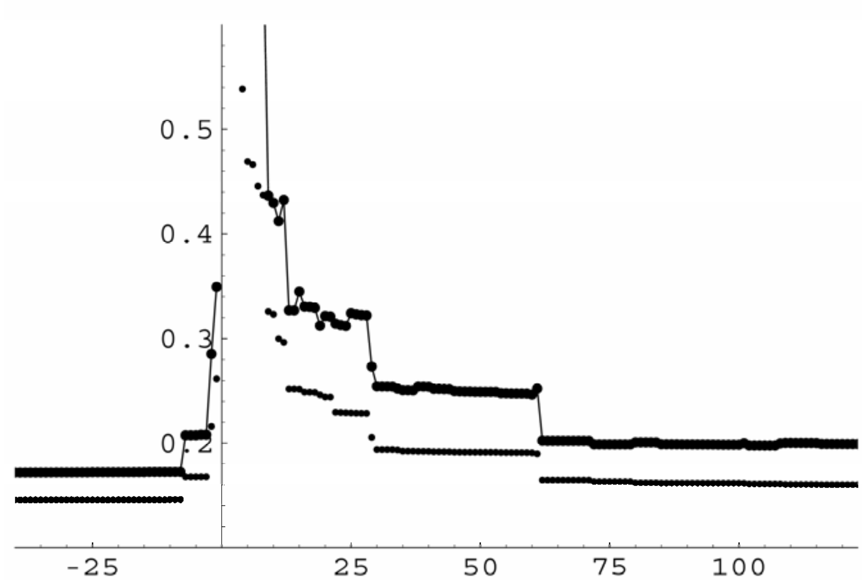

Figure 3. The normalized regularized bias $\delta'(x, 13)/\sqrt{x}$ versus the Chebyshev's bias $\delta(x, 13)$ at the prime champions of $\delta(x, 13)$ (when $\delta(x, 13) > 0$), and the curve at the prime champions of $-\delta(x, 13)$ (when $\delta(x, 13) < 0$). The extremal prime champions in the plot are $x = 263881$ (with $\delta = 123$) and $x = 905761$ (with $\delta = -40$). The small dot curve corresponds to the fit of $\delta'(x, 13)/\sqrt{x}$ by $2/\log x$ calculated in Section 3.

Finally, for the modulus $q = 11$, the imaginary part of the first zero is quite small, $\gamma_1 \sim 0.209$, and the variance is high, $V(\kappa_{-11}) \sim 0.507$. In such a case, as shown in Figure 4, the approximation of the regularized bias by $2/\log x$ is good in the whole range of values of x.

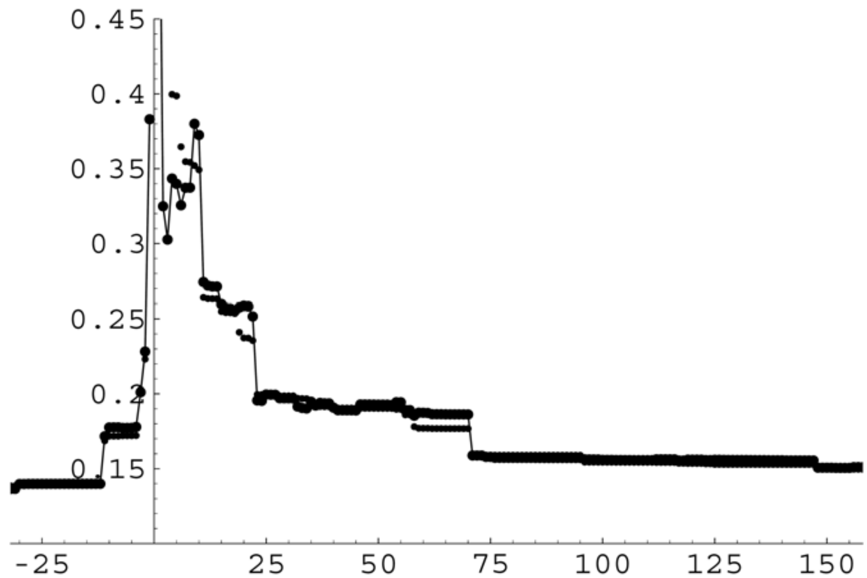

Figure 4. The normalized regularized bias $\delta'(x, 11)/\sqrt{x}$ versus the Chebyshev's bias $\delta(x, 11)$ at the prime champions of $\delta(x, 11)$ (when $\delta(x, 11) > 0$), and the curve at the prime champions of $-\delta(x, 11)$ (when $\delta(x, 11) < 0$). The extremal prime champions in the plot are $x = 638567$ (with $\delta = 158$) and $x = 1867321$ (with $\delta = -32$). The small dot curve corresponds to the (very good) fit of $\delta'(x, 11)/\sqrt{x}$ by $2/\log x$ calculated in Section 3.

3. Tentative Proof of Proposition 1.1

For approaching the Proposition 1.1, we reformulate it in a simpler way as

Proposition 3.1. *One introduces the regularized counting function* $\pi'(x; q, l) := \pi(x; q, l) - \psi(x; q, l)/\log x$. *The statement* $\pi'(x; q, N) > \pi'(x; q, R), \forall x$ (ii), *is equivalent to GRH for the modulus* q.

Tentative Proof 3.2. First observe that Proposition 1.1 follows from Proposition 3.1. This is straightforward because according to [12, p. 260], the prime number theorem for arithmetic progressions leads to the approximation

$$\mathrm{li}[\phi(q)\psi(x;\, q,\, l)] \sim \mathrm{li}(x) + \frac{\phi(q)\psi(x;\, q,\, l) - x}{\log x}. \tag{3.1}$$

As a result,

$$\begin{aligned}\delta'(x,\, q) &= B(x;\, q,\, R) - B(x;\, q,\, N)\\ &= \mathrm{li}[\phi(q)\psi(x;\, q,\, R)] - \mathrm{li}[\phi(q)\psi(x;\, q,\, N)] + \phi(q)\delta(x,\, q)\\ &\sim \phi(q)[\pi'(x;\, q,\, N) - \pi'(x;\, q,\, R)].\end{aligned}$$

The asymptotic equivalence in (3.1) holds up to the error term [12, p. 260] $O(\frac{R(x)}{x\log x})$, with

$$R(x) = \min\left(x^{\theta_q}\log^2 x,\; xe^{-a\sqrt{\log x}}\right), \quad a > 0,$$

$$\theta_q = \max_{\kappa \mod q}(\sup \Re(\rho),\; \rho \text{ a zero of } L(s,\, \kappa)).$$

Let us now look at the statement GRH $\Rightarrow$ (i). Following [13, p. 178-179], one has

$$\psi(x;\, q,\, a) = \frac{1}{\phi(q)}\sum_{\kappa \mod q}\bar{\kappa}(a)\psi(x,\, \kappa),$$

and under GRH,

$$\pi(x;\, q,\, a) = \frac{\pi(x)}{\phi(q)} - \frac{c(q,\, a)}{\phi(q)}\frac{\sqrt{x}}{\log x} + \frac{1}{\phi(q)\log x}\sum_{\kappa \neq \kappa_0}\bar{\kappa}(a)\psi(x,\, \kappa) + O(\frac{\sqrt{x}}{\log^2 x}),$$

where κ_0 is the principal character modulo q and

$$c(q,\, a) = -1 + \#\{1 \le b \le q : b^2 = a \mod q\},$$

for coprimes integers a and q. Note that for an odd prime $q = p$, one has $c(p,\, a) = \left(\frac{a}{p}\right)$.

Thus, under GRH,

$$\pi(x;\, q,\, N) - \pi(x;\, q,\, R) = \frac{1}{\phi(q)\log x}\Big[\sqrt{x}(c(q,\, R) - c(q,\, N))$$

$$+ \sum\nolimits_{\kappa \bmod q} (\overline{\kappa}(N) - \overline{\kappa}(R))\psi(x,\, \kappa)$$

$$+ O\left(\frac{\sqrt{x}}{\log^2 x}\right)\Big]. \tag{3.2}$$

The sum could be taken over all characters because $\overline{\kappa}_0(N) = \overline{\kappa}_0(R)$. In addition, we have

$$\psi(x;\, q,\, N) - \psi(x;\, q,\, R) = \frac{1}{\phi(q)} \sum_{\kappa \bmod q} [\overline{\kappa}(N) - \overline{\kappa}(R)]\psi(x,\, \kappa). \tag{3.3}$$

Using (3.2) and (3.3), the regularized bias reads

$$\delta'(x,\, q) \sim \pi'(x;\, q,\, N) - \pi'(x;\, q,\, R)$$

$$= \frac{\sqrt{x}}{\log x}[c(q,\, R) - c(q,\, N)] + O\left(\frac{\sqrt{x}}{\log^2 x}\right). \tag{3.4}$$

For the modulus $q = 4$, we have $c(q,\, 1) = -1 + 2 = 1$ and $c(q,\, 3) = -1$ so that $\delta'(x,\, 4) = \frac{2\sqrt{x}}{\log x}$. The same result is obtained for a prime modulus $q = p$ since $c(p,\, N) = -1$ and $c(p,\, R) = c(p,\, 1) = \left(\frac{1}{p}\right) = 1$.

For x large enough and under GRH for the modulus q (at least for $q = 4$ and for a prime modulus $q = p$), the regularized bias $\delta'(x,\, q)$ is positive and one has the inequality $\pi'(x;\, q,\, N) > \pi'(x;\, q,\, R)$. Besides, for (numerically reachable) small values of x, we found in Section 2 that $\delta'(x,\, q) > 0$ (at least for a few selected values of q). This strengthens our conviction of the non-negativity of $\delta'(x,\, q)$ for all moduli. If GRH does not hold, then using [12, Lemma 2], one has

$$B(x;\, q,\, a) = \Omega_{\pm}(x^{\xi}) \text{ for any } \xi < \theta_q.$$

Applying this asymptotic result to the residue classes $a = R$ and $a = N$, there exist infinitely many values $x = x_1$ and $x = x_2$ satisfying

$$B(x_1;\, q,\, R) < -x_1^{\xi} \text{ and } B(x_2;\, q,\, N) > x_2^{\xi} \text{ for any } \xi < \theta_q,$$

so that one obtains

$$B(x_1;\, q,\, R) - B(x_2;\, q,\, N) < -x_1^{\xi} - x_2^{\xi} < 0. \tag{3.5}$$

Selecting a pair $(x_1,\, x_2)$ either

$$B(x_1;\, q,\, R) > B(x_2;\, q,\, R),$$

so that $B(x_2;\, q,\, R) - B(x_2;\, q,\, N) < 0$ and (i) is violated at x_2, or

$$B(x_1;\, q,\, R) < B(x_2;\, q,\, R). \tag{3.6}$$

In the last case, either $B(x_1;\, q,\, N) > B(x_2;\, q,\, N)$, so that $B(x_1;\, q,\, R) - B(x_1;\, q,\, N) < 0$ and the inequality (i) is violated at x_1, or simultaneously,

$$B(x_1;\, q,\, N) < B(x_2;\, q,\, N) \text{ and } B(x_1;\, q,\, R) < B(x_2;\, q,\, R),$$

which implies (3.5) and the violation of (i) at $x = x_1 = x_2$.

To finalize the proof of 3.1, and simultaneously that of 1.1, one makes use of the asymptotic equivalence of (i) and (ii), that is, if GRH is true $\Rightarrow$ (ii) $\Rightarrow$ (i), and if GRH is wrong, (i) may be violated and (ii) as well.

Then, Proposition 2.1 also follows as a straigthforward consequence of Proposition 1.1.

4. Summary

We have found that the asymmetry in the prime counting function $\pi(x;\, q,\, a)$ between the quadratic residues $a = R$ and the quadratic nonresidues $a = N$ for the modulus q can be encoded in the function $B(x;\, q,\, a)$ [defined in (1.11)] introduced by Robin the context of GRH [12], or into the regularized prime counting function $\pi'(x;\, q,\, a)$ as in Proposition 3.1. The bias in π' reflects the bias in π conditionaly under

GRH for the modulus q. Our conjecture has been initiated by detailed computer calculations presented in Section 2 and tentatively proved in Section 3. Further work could follow the work about the connection of π, and thus of π', to the sum of squares function $r_2(n)$ [10].

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